On the Validity of Covariance and Variogram Functions on the Sphere

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Abstract. In this article, we study the validity of commonly used covariance functions on the sphere. In particular, we show that both the exponential and spherical covariance functions are valid. However, the two radon transforms of exponential models (order 2 and 4) and the Guassian covariance function are not valid. In addition, variograms on the sphere are defined and investigated. A necessary and sufficient condition for variogram functions to be valid on the sphere is derived.

KEY WORDS: Exponential covariance, Spherical covariance, Necessary and sufficient condition, Brownian motion

1. Introduction

Global-scale processes and phenomena are of utmost importance in the geosciences. Data from global networks of in situ and satellite sensors are used to monitor a wide array of geophysical processes. Most methods and models in spatial statistics are developed in Euclidean spaces R^n and these methods have not been investigated as intensively in spherical cooridnate systems (Robeson, 1997). Alkhaled et al. (2008), for instance, use a spherical version of the exponential semivariogram model to study global measurements of carbon dioxide concentration. To adapt Euclidean approaches to processes on the sphere, however, one must first evaluate their properties to ensure their admissibility.

Consider a random process $X(p), p \in S^2$, where S^2 is a unit sphere in R^3, and the location $p = (\phi, \lambda)$ with latitude ϕ and longitude λ, $0 \leq \phi \leq \pi, 0 \leq \lambda < 2\pi$. The process is stationary if the covariance function $\text{cov}(X(p_1), X(p_2))$ depends solely on the spherical angle $\theta(p_1, p_2) \in [0, \pi]$, where

$$\cos(\theta(p_1, p_2)) = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos(\lambda_1 - \lambda_2).$$

A real continuous function $C(\cdot)$ is said to be a valid covariance function on the sphere if and only if it is of the form (Schoenberg, 1942, Yaglom, 1987)

$$C(\theta) = \sum_{n=0}^{\infty} b_n P_n(\cos\theta), \quad b_n \geq 0, \quad \sum_{n=0}^{\infty} b_n < \infty, \quad \theta \in [0, \pi], \tag{1}$$

where $P_n(\cdot)$ are the Legendre polynomials. This result can be used to check the validity of commonly used functions. In Section 2, we will show that both exponential and spherical covariance functions are valid on the sphere, while some counterexamples are presented to show that a valid covariance function in Euclidean space may not be admissible on the sphere. In practice, many applications focus on intrinsically stationary processes and their variograms. While such processes have been well studied in Euclidean space (Cressie, 1993, Chiles and Delfiner, 1999), the extension to the sphere needs more investigation. In Section 3, we give a necessary and sufficient condition for a continuous function to be valid, namely, conditionally negative definite. An example of Brownian motion on the sphere is presented to demonstrate that an intrinsically stationary process may not be stationary. This answers some questions raised in Reguzzoni et al. (2005).

2. Validity of covariance functions

A real continuous function $C(\cdot)$ is said to be a valid covariance function if it is of the form (1). By orthogonality of the Legendre polynomials (equation 7.112.1, Gradshteyn and Ryzhik, 1994), the coefficients

$$b_n = \frac{2n+1}{2} \int_0^{\pi} C(\theta) P_n(\cos\theta) \sin\theta d\theta, \quad n = 0, 1, 2, \ldots. \tag{2}$$

Therefore, we can investigate positive definiteness of $C(\cdot)$ by checking whether b_n is non-negative and $\sum_n b_n < \infty$ (Theorem 1, Schoenberg, 1942). In this section, the validity of several covariance functions are evaluated.

Example 1: Exponential Model. The exponential covariance function with scale parameter $a > 0$ is defined by

$$C(\theta) = e^{-\theta/a} \qquad \text{for } 0 \leq \theta \leq \pi.$$

This model is a valid covariance in R^n for any n (see Yaglom, 1987, pages 362-363). To check whether it is valid on the sphere, we compute the coefficients in (2) through the following sine expansion of the Legendre polynomials (Hobson, 1993, page 20, also equation 8.826.1, Gradshteyn and Ryzhik, 1994):

$$\frac{\pi}{4} P_n(\cos\theta) = \frac{2 \cdot 4 \cdots 2n}{3 \cdot 5 \ldots (2n+1)} \left\{ \sin(n+1)\theta + \frac{1}{1} \frac{n+1}{(2n+3)} \sin(n+3)\theta + \cdots \right\}, \tag{3}$$

or more compactly,

$$P_n(\cos\theta) := \sum_{k=0}^{\infty} D_{nk} \sin(n+2k+1)\theta, \quad \text{for } 0 < \theta < \pi,$$

where

$$D_{nk} = \frac{4}{\pi} \frac{2 \cdot 4 \cdots 2n}{3 \cdot 5 \cdots (2n+1)} \frac{1 \cdot 3 \cdots (2k-1) \cdot (n+1) \cdots (n+k)}{1 \cdot 2 \cdots k \cdot (2n+3) \cdots (2n+2k+1)} > 0,$$

representing the coefficients corresponding to $\sin(n+2k+1)\theta$ in the expansion. Hence,

$$\begin{aligned} b_n &= \frac{2n+1}{2} \int_0^\pi C(\theta) \sum_{k=0}^{\infty} D_{nk} \sin(n+2k+1)\theta \sin\theta d\theta \\ &= \sum_{k=0}^{\infty} \frac{2n+1}{2} D_{nk} \int_0^\pi C(\theta) \sin(n+2k+1)\theta \sin\theta d\theta \\ &:= \sum_{k=0}^{\infty} \frac{2n+1}{2} D_{nk} a_{nk}, \end{aligned}$$

where

$$\begin{aligned} a_{nk} &= \int_0^\pi C(\theta) \sin(n+2k+1)\theta \sin\theta d\theta \\ &= \int_0^\pi C(\theta) \frac{1}{2} [\cos(n+2k)\theta - \cos(n+2k+2)\theta] d\theta \\ &= \frac{1}{2} \int_0^\pi e^{-\theta/a} [\cos(n+2k)\theta - \cos(n+2k+2)\theta] d\theta \\ &= \begin{cases} \frac{a(1-e^{-\pi/a})}{2} \left(\frac{1}{1+(n+2k)^2 a^2} - \frac{1}{1+(n+2k+2)^2 a^2} \right) > 0, & n \text{ even}, \\ \frac{a(1+e^{-\pi/a})}{2} \left(\frac{1}{1+(n+2k)^2 a^2} - \frac{1}{1+(n+2k+2)^2 a^2} \right) > 0, & n \text{ odd}, \end{cases} \end{aligned}$$

implying that $b_n \geq 0$. Now it suffices to prove that $\sum_n b_n < \infty$. First, for each fixed $a > 0$ and some constant M_0 free of n and k, it is clear that

$$a_{nk} \leq \frac{M_0}{(n+2k)^3}.$$

In addition, we have,

$$\frac{1\cdot 3\cdots(2k-1)\cdot(n+1)\cdots(n+k)}{1\cdot 2\cdots k\cdot(2n+3)\cdots(2n+2k+1)}<1.$$

Let $A_n \sim B_n$ of two real sequences A_n and B_n represent $\lim_{n\to\infty}\frac{A_n}{B_n}\leq M$ for some constant $M>0$. Thus, for sufficiently large n, by Sterling's approximation to factorials and noting that $(1+1/(2n+1))^{2n+1}\to e$, we have

$$D_{nk} \;\leq\; \frac{4}{\pi}\frac{2\cdot 4\cdots 2n}{3\cdot 5\cdots(2n+1)}=\frac{4}{\pi}\frac{(2^n n!)\cdot(2^n n!)}{(2n+1)!}\sim\frac{4}{\pi}\frac{2^{2n}\frac{2\pi}{n+1}\frac{(n+1)^{2n+2}}{e^{2n+2}}}{\sqrt{\frac{2\pi}{2n+2}}\frac{(2n+2)^{2n+2}}{e^{2n+2}}}\sim\frac{1}{\sqrt{n+1}}.$$

Hence, for sufficiently large n,

$$b_n=\sum_{k=0}^{\infty}\frac{2n+1}{2}D_{nk}a_{nk}\sim\frac{2n+1}{\sqrt{n+1}}\sum_{k=0}^{\infty}\frac{1}{(n+2k)^3}\sim\frac{2n+1}{\sqrt{n+1}}\frac{1}{n^2}\sim\frac{1}{n^{3/2}}.$$

That is, b_n goes to zero at the rate of $n^{-3/2}$ when $n\to\infty$. Therefore, $\sum b_n<\infty$, we conclude that the exponential covariance function is valid on the sphere.

Example 2: Spherical model. A spherical covariance function has the form

$$C(\theta)=\begin{cases}1-\frac{3\theta}{2a}+\frac{1}{2}\frac{\theta^3}{a^3}, & \text{if } \theta\leq a,\\ 0, & \text{if } \theta>a,\end{cases}$$

where $a>0$ is a scale parameter. Using the same sine expansion (3), we first show that $a_{nk}\geq 0$, where

$$\begin{aligned}a_{nk}&=\int_0^{\pi}C(\theta)\sin(n+k)\theta\sin\theta d\theta\\ &=\int_0^{a}\left(1-\frac{3\theta}{2a}+\frac{\theta^3}{2a^3}\right)\frac{1}{2}[\cos(n+k-1)\theta-\cos(n+k+1)\theta]d\theta\\ &:=\frac{1}{2}(g(n+k-1)-g(n+k+1)).\end{aligned}$$

Here, we define, for $x>0$,

$$g(x)=\int_0^{a}\left(1-\frac{3\theta}{2a}+\frac{\theta^3}{2a^3}\right)\cos x\theta d\theta.$$

Note that the integration above yields

$$g(x) = -\frac{3}{2x^2 a} + \frac{3}{x^4 a^3} - \frac{3}{x^3 a^2}\sin ax - \frac{3}{x^4 a^3}\cos ax.$$

By taking the first derivative with respect to x, we have, for $x > 0$,

$$\begin{aligned} g'(x) &= -\frac{3}{x^3 a} - \frac{12}{x^5 a^3} + \frac{9}{x^4 a^2}\sin ax - \frac{3}{x^3 a}\cos ax + \frac{12}{x^5 a^3}\cos ax + \frac{3}{x^4 a^2}\sin ax \\ &= -\frac{6}{x^5 a^3}\left[4\sin^2\frac{ax}{2} - 4ax\sin\frac{ax}{2}\cos\frac{ax}{2} + x^2 a^2\cos^2\frac{ax}{2}\right] \\ &= -\frac{6}{x^5 a^3}\left(2\sin\frac{ax}{2} - ax\cos\frac{ax}{2}\right)^2 \le 0. \end{aligned}$$

Hence, $g(x), x > 0$ is an non-increasing function, implying that

$$a_{nk} = \frac{1}{2}[g(n+k-1) - g(n+k+1)] \ge 0.$$

That is, $b_n \ge 0$ for all n. The proof of $\sum_n b_n < \infty$ in the exponential model can be easily adapted to show that $\sum_n b_n < \infty$ in the spherical model. Therefore, the spherical covariance function is also valid on the sphere.

Example 3: Radon Transform of Exponential Models. The following two radon transforms of exponential models (order 2 and 4) can be found in Chiles and Delfiner (1999, page 85) and Gneiting (1999):

$$C_1(\theta) = e^{-\theta/a}(1 + \theta/a),$$

and

$$C_2(\theta) = e^{-\theta/a}(1 + \theta/a + \theta^2/3a^2), .$$

By direct computation, we found that $b_6 < 0$ for $C_1(\theta)$, and $b_2 < 0$ for $C_2(\theta)$, when $a = 1$. Therefore, neither is the valid covariance function on the sphere.

Remarks. 1. The Gaussian covariance function $C(\theta) = \exp\{-a\theta^2\}$ is shown to be invalid in Gneiting (1999). Applying the method in this section, a direct computation yields $b_8 < 0$, which also indicates that the Gaussian covariance function is not valid on the sphere.

2. Note that if a function $C_0(h)$ is a valid covariance function in R^3, then the function

$$C(\theta) = C_0(2\sin(\theta/2))$$

is a valid covariance function on unit sphere S^2. In general, any function of the form

$$C(\theta) = \int_0^\infty J_0(2\omega \sin(\theta/2)) d\Phi(\omega),$$

where $J_0(\cdot)$ is a Bessel function, and $\Phi(\cdot)$ is a bounded nondecreasing function, is valid on a unit sphere S^2 (see Yadrendo, 1983, page 76, or Yalgom, 1987, page 389). Therefore, one can obtain a rich family of valid covariance functions on the sphere.

3. When a function is not positive definite, negative variances can occur, for example, see Armstrong and Jabin (1981).

4. We use sine expansion for both exponential and spherical covariance functions to show that $b_n \geq 0$ and $\sum_n b_n < \infty$. The expansion is an infinite summation. In fact a closed form expression with finite summation for b_n can be obtained through a cosine expansion of the Legendre polynomials, see more details in Appendix B. Note that the cosine expansion in Gradshteyn and Ryzhik (1994), equation (8.911.4) is not completely accurate, we present a correct formula of the cosine expansion in Appendix B.

3. Variograms on the sphere

Similar to the intrinsically stationary process on R^n (Cressie, 1993), we have

Definition. Suppose $\{X(p) : p \in S^2\}$ satisfies

$$\mathrm{E}(X(p)) = \mu, \quad \text{for all} \quad p \in S^2,$$

and

$$\text{var}(X(p_1) - X(p_2)) = 2\gamma(\theta(p_1, p_2)), \quad \text{for all} \quad p_1, p_2 \in S^2,$$

then $X(\cdot)$ is said to be intrinsically stationary. The quantity $2\gamma(\cdot)$ is called the variogram, and $\gamma(\cdot)$ is the semi-variogram.

Proposition 1. The variogram $2\gamma(\cdot)$ is conditionally negative definite, namely,

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j 2\gamma(\theta(p_i, p_j)) \leq 0,$$

for any finite number of spatial locations $\{p_i, i = 1, \ldots, m\}$ on S^2 and real numbers $\{a_i, i = 1, \ldots, m\}$ satisfying $\sum_{i=1}^{m} a_i = 0$.

Proof. Consider an intrinsically stationary process $X(p), p \in S^2$. Since $\sum_{i=1}^{m} a_i = 0$, we have

$$\left\{\sum_{i=1}^{m} a_i X(p_i)\right\}^2 = -\frac{1}{2}\left\{\sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j (X(p_i) - X(p_j))^2\right\}.$$

Taking expectation both sides,

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j 2\gamma(\theta(p_i, p_j)) = -2\text{E}\left\{\sum_{i=1}^{m} a_i X(p_i)\right\}^2 \leq 0.$$

◇

To investigate whether a function is conditionally negative definite on the sphere, we obtain the following theorem.

Theorem 1. A continuous function $2\gamma(\cdot)$ satisfying $\gamma(0) = 0$ is conditionally negative definite if and only if $2\gamma(\cdot)$ is of the form

$$2\gamma(\theta) = \sum_{n=1}^{\infty} b_n(1 - P_n(\cos\theta)), \quad b_n \geq 0, \quad \sum_{n=1}^{\infty} b_n < \infty, \tag{4}$$

and $P_n(\cdot)$ are the Legendre polynomials.

Proof. In order to prove the direct part of the theorem, let $2\gamma(\cdot)$ be of the form of (4), then

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j 2\gamma(\theta(p_i, p_j)) = -\sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j \sum_{n=1}^{\infty} b_n P_n(\cos(\theta(p_i, p_j))) \leq 0.$$

This is because $\sum_{i=1}^{m} a_i = 0$, and Schoenberg (1942, Theorem 1).

The converse part is proved as follows. The function $2\gamma(\cdot)$ is conditionally negative definite if and only if $e^{-\lambda\gamma(\cdot)}$ is positive definite (Schoenberg, 1938). Therefore, the function

$$\Phi(\theta, \lambda) = \frac{1}{\lambda}(1 - \exp\{-\lambda\gamma(\theta)\}) = \sum_{n=0}^{\infty} \lambda^{-1} c_n(\lambda)(1 - P_n(\cos\theta)).$$

However, $\Phi(\theta, \lambda) \to \gamma(\theta)$ as $\lambda \to 0$, and $\Phi(\theta, \lambda)$ is a function of class K (see Appendix A). As the limit function $\gamma(\cdot)$ is continuous throughout $[0, \pi]$, Lemma 2 in Appendix A implies that $\gamma(\cdot)$ is of the desired form (4). ♢

Remarks. 1. The proof of Theorem 1 is in the similar line of Theorem 3 in Schoenberg (1942), where the sphere of infinite dimension S^{∞} is considered. Note that Lemma 3 in Schoenberg (1942) is not applicable, we present the new lemmas in Appendix A along with the proof. A general result can be found in Bochner (1941) and Schoenberg (1942, note 12).

2. When a process is stationary, the covariance function $C(\theta)$ adapts the form of $\sum_{n=0}^{\infty} b_n P_n(\cos\theta)$, (see Schoenberg, 1941). It is clear that

$$2\gamma(\theta) = 2C(0) - 2C(\theta) = \sum_{n=1}^{\infty} 2b_n(1 - P_n(\cos\theta)).$$

3. If a variogram is continuous and has first derivative, the coefficients b_n in (4) are

$$b_n = -\frac{2n+1}{2n(n+1)} \int_{-1}^{1} \gamma(\cos^{-1} x) \frac{d}{dx} P_n^1(x) \sqrt{1 - x^2}, \quad (5)$$

where $x = \cos\theta$. This can be used to check the admissibility of an arbitrary function on the sphere (see next remark).

4. The stationary process is intrinsically stationary. The converse is not true. Consider a Brownian motion $X(p)$ on S^2 (Gangolli, 1967) that

$$\mathrm{cov}(X(p_1), X(p_2)) = \frac{1}{2}(\theta(p_1, p_0) + \theta(p_2, p_0) - \theta(p_1, p_2)), \quad p_1, p_2 \in S^2,$$

and p_0 is a fixed arbitrary point (for example, north pole). It is clear that this process is not stationary, however, the variance of $X(p_1) - X(p_2)$

$$\mathrm{var}(X(p_1) - X(p_2)) = \theta(p_1, p_2),$$

depends solely on $\theta(p_1, p_2)$. That is, the process $X(p)$ is intrinsically stationary, but not stationary. Furthermore, by using (5), one can show that, for $m = 0, 1, \ldots,$

$$b_{2m} = 0, \quad b_{2m+1} = \frac{4m+3}{4(m+1)(2m+1)} \frac{\Gamma(m+1/2)\Gamma(m+3/2)}{m!(m+1)!},$$

which are all non-negative. To see that $\sum_{n=0}^{\infty} b_n < \infty$, we use Sterling's approximation to Gamma function and obtain,

$$\begin{aligned} b_{2m+1} &\sim \frac{1}{2m+1} \frac{\left(m+\frac{1}{2}\right)^{m+\frac{1}{2}} \left(m+\frac{3}{2}\right)^{m+\frac{3}{2}}}{(m+1)^{m+1} (m+2)^{m+2}} \\ &\sim \frac{1}{(2m+1)^2} \left(\frac{m+\frac{1}{2}}{m+1}\right)^{m+\frac{1}{2}} \left(\frac{m+\frac{3}{2}}{m+2}\right)^{m+\frac{3}{2}} \\ &\sim \frac{1}{(2m+1)^2}, \text{ for } m \text{ large.} \end{aligned}$$

Therefore, $\sum_{n=0}^{\infty} b_n < \infty$.

5. In practice, one may use weighted least squares to estimate the parameters (Cressie, 1985) for valid covariance and variogram models on the sphere.

6. By Remark 2, the variogram can be derived from the covariance function $\gamma(\theta) = C(0) - C(\theta)$. Therefore, the results we obtain in Section 2 also apply to the associated variogram models for stationary processes. For example, both the exponential and spherical variograms are valid, not only

in Euclidean space but also on S^2. The variogram models for the Gaussian covariance function and the two radon transforms of the exponential models, however, are not valid on S^2.

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Appendix A

We present the lemmas needed for the proof of Theorem 1 and the definition of class K in this Appendix. First, let K denote the class of functions $\phi(x)$ of the form

$$\phi(x) = \sum_{n=1}^{\infty} a_n(1 - P_n(x)), \quad a_n \geq 0, \quad -1 \leq x \leq 1.$$

In addition, suppose

$$\psi(x) = \sum_{\nu=1}^{\infty} c_\nu P'_\nu(x), \quad c_\nu \geq 0, \quad -1 < x < 1.$$

such that

$$\int_0^{1-0} \psi(x)dx \quad \text{exists},$$

and we let the class K_1 be the functions of the form

$$K_1 = \left\{ \phi(x) | \phi(x) = \int_x^{1-0} \psi(x)dx, -1 \le x < 1, \phi(1) = 0 \right\}.$$

We have the following lemma.

Lemma 1. $K = K_1$.

Proof. For " $K \subset K_1$". From $\phi(x) = \sum_{n=1}^{\infty} a_n(1 - P_n(x))$,

$$-\phi'(x) = -\sum_{n=1}^{\infty}(-a_n)P_n'(x) = \psi(x).$$

for $-1 < x < 1$. Hence,

$$\phi(x) - \phi(1-\epsilon) = \int_x^{1-\epsilon} \psi(x)dx.$$

Letting $\epsilon \to 0$,

$$\phi(x) = \int_x^{1-0} \psi(x)dx,$$

therefore, $\phi(x) \in K_1$.

For "$K_1 \subset K$". Suppose $\phi(x) \in K_1$,

$$\phi(x) - \phi(1-\epsilon) = \int_x^{1-\epsilon} \sum_{\nu=1}^{\infty} c_\nu P_\nu'(x)dx = \sum_{\nu=1}^{\infty} c_\nu [P_\nu(1-\epsilon) - P_\nu(x)],$$

Note that $P_\nu(1) = 1$, and $\phi(1) = 1$. Letting $\epsilon \to 0$,

$$\phi(x) = \sum_{\nu=1}^{\infty} c_\nu [1 - P_\nu(x)] \in K.$$

$\diamondsuit$

Now we present the main lemma used for Theorem 1.

Lemma 2. If $\{\phi_n(x)\}$ is a sequence of functions in the class K and

$$\lim_{n\to\infty} \phi_n(x) = \phi_0(x), \quad -1 \le x \le 1,$$

where $\phi_0(x)$ is continuous in $-1 \le x \le 1$, then also $\phi_0(x) \in K$.

Proof. Let

$$\phi_n(x) = \sum_{\nu=1}^{\infty} a_{n\nu}(1 - P_\nu(x)), \qquad \text{for } a_{n\nu} \geq 0.$$

By (8.917.4) of Gradshteyn and Ryzhik (1994),

$$|P_\nu(\cos\phi)| \leq \frac{1}{\sqrt{\nu \sin\phi}}, \quad \text{or} \quad |P_\nu(x)| \leq \frac{1}{\sqrt{\nu}(1-x^2)^{1/4}}$$

For $|x| \leq r < 0$,

$$|P_\nu(x)| \leq \frac{1}{(1-r^2)^{1/4}}.$$

Therefore,

$$-\frac{1}{(1-r^2)^{1/4}} \leq P_\nu(x) \leq \frac{1}{(1-r^2)^{1/4}}.$$

That is,

$$0 \leq 1 - P_\nu(x) \leq 1 + \frac{1}{(1-r^2)^{1/4}}.$$

Therefore, as $a_{n\nu} \geq 0$,

$$0 \leq \phi_n(x) = \sum_{\nu=1}^{\infty} a_{n\nu}(1 - P_\nu(x)) \leq \left(1 + \frac{1}{(1-r^2)^{1/4}}\right) \sum_{\nu=1}^{\infty} a_{n\nu}$$

To show $\phi_n(x)$ is uniformly bounded, note that for a fixed $r < 1$, there exists a n_0 such that

$$1 - \frac{1}{\sqrt{\nu}} \frac{1}{(1-r^2)^{1/4}} > M_1 > 0, \quad \text{for all} \quad \nu > n_0.$$

That is, for $\nu > n_0$,

$$1 - P_\nu(r) \geq 1 - \frac{1}{\sqrt{\nu}} \frac{1}{(1-r^2)^{1/4}} > M_1.$$

Therefore,

$$\phi_n(r) = \sum_{n=1}^{n_0} a_{n\nu}(1 - P_\nu(r)) + \sum_{n=n_0+1}^{\infty} a_{n\nu}(1 - P_\nu(r)).$$

For $\nu \leq n_0$, there exists M_2 such that

$$1 - P_\nu(r) \geq M_2 > 0.$$

We have

$$\phi_n(r) \geq \sum_{n=1}^{n_0} a_{n\nu} M_2 + \sum_{n=n_0+1}^{\infty} a_{n\nu} M_1 \geq \min(M_1, M_2) \sum a_{n\nu}.$$

Which leads to

$$\sum a_{n\nu} \leq C\phi_n(r),$$

where $C = 1/\min(M_1, M_2) > 0$. Therefore, $\phi_n(x)$ are uniformly bounded in every circle $|x| \leq r, (r < 1)$. By the Vitali-Porter convergence theorem, $\phi_n(x)$ converge uniformly inside $|x| < 1$ and that $\phi_0(x)$ is analytic and regular in $|x| < 1$, also $\phi_n' \to \phi_0'$ uniformly inside $|x| < 1$. (See, Theorem 2.1, page 151 of Conway, 1978).

Note that

$$\phi_n'(x) = (-1)\sum_{\nu=1}^{\infty} a_{n\nu} P_\nu'(x)$$

that is,

$$(-1)\sqrt{1-x^2}\phi_n'(x) = \sum_{\nu=1}^{\infty} a_{n\nu}\sqrt{1-x^2}P_\nu'(x) = \sum_{\nu=1}^{\infty} a_{n\nu}(-1)P_\nu^1(x)$$

(by the definition of the associated Legendre's polynomial $P_\nu^1(x) = (-1)\sqrt{1-x^2}P_\nu'(x)$). It is clear that

$$(-1)\frac{2\nu(\nu+1)}{2\nu+1}a_{n\nu} = \int_{-1}^{1}(-1)\sqrt{1-x^2}\phi_n'(x)P_\nu^1(x)dx, \tag{6}$$

due to the orthogonality of the associated Legendre's polynomials (for example, 7.112.1 from Gradshteyn and Ryzhik, 1994), *i.e.,*

$$\int_{-1}^{1} P_{\nu_1}^1(x)P_{\nu_2}^1(x)dx = \begin{cases} \frac{2(\nu_1+1)\nu_1}{2\nu_1+1} & \text{if } \nu_1 = \nu_2, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Next, we prove

$$\int_{-1}^{1}(-1)\sqrt{1-x^2}\phi_n'(x)P_\nu^1(x)dx \to \int_{-1}^{1}(-1)\sqrt{1-x^2}\phi_0'(x)P_\nu^1(x)dx, \tag{8}$$

as $n \to \infty$. Note that, from integration by parts,

$$\begin{aligned}
\int_{-1}^{1}(-1)\sqrt{1-x^2}\phi_n'(x)P_\nu^1(x)dx &= \int_{-1}^{1}(-1)\sqrt{1-x^2}P_\nu^1(x)d\phi_n(x) \\
&= (-1)\sqrt{1-x^2}P_\nu^1(x)\phi_n(x)|_{-1}^{1} - \int_{-1}^{1}\phi_n(x)\frac{d}{dx}\left((-1)\sqrt{1-x^2}P_\nu^1(x)\right)dx \\
&:= \int_{-1}^{1}\phi_n(x)g(x)dx, \qquad \text{and similarly,} \\
\int_{-1}^{1}(-1)\sqrt{1-x^2}\phi_0'(x)P_\nu^1(x)dx &= \int_{-1}^{1}\phi_0(x)g(x)dx
\end{aligned}$$

where $g(x) = \frac{d}{dx}\sqrt{1-x^2}P_\nu^1(x)$. Therefore, to prove (8), it suffices to prove that

$$\int_{-1}^{1} \phi_n(x)g(x)dx \to \int_{-1}^{1} \phi_0(x)g(x)dx. \tag{9}$$

First, it is clear that $\sqrt{1-x^2}P_\nu^1(x) = (-1)(1-x^2)P_\nu'(x)$ is a polynomial on $[-1,1]$, which can be bounded by some $M > 0$. By assumption, $\phi_0(x)$ is a continuous function on $[-1,1]$, and so $\phi_0(x)$ is a bounded function on $[-1,1]$. Hence, $\phi_n(x)$ is bounded on $[-1,1]$ since $\lim_{n\to\infty}\phi_n(x) = \phi_0(x)$. Therefore, (9) holds by the dominate convergence theorem.

Notice that $\sqrt{1-x^2}\phi_0'(x)$ can be expressed as

$$\sqrt{1-x^2}\phi_0'(x) = \sum_{\nu=1}^{\infty} c_\nu P_\nu^1(x),$$

for some c_ν since the family $\{P_\nu^1(x)\}_\nu$ is an orthogonal basis for any continuous function on $[-1,1]$. By equations (6), (7), and (8), for each $\nu > 0$,

$$c_\nu = \lim_{n\to\infty} a_{n\nu},$$

implying that $c_\nu \geq 0$. With $P_\nu^1(x) = (-1)\sqrt{1-x^2}P_\nu'(x)$, we have

$$-\phi_0'(x) = \sum c_\nu P_\nu'(x), \quad \text{and} \quad \psi_0(x) = -\phi_0'(x) = \sum c_\nu P_\nu'(x),$$

implying

$$\phi_0(x) - \phi_0(1-\epsilon) = \int_x^{1-\epsilon} \psi_0(x)dx.$$

Since $\phi_0(1-0) = \phi_0(1) = 0$, by letting $\epsilon \to 0$,

$$\phi_0(x) = \int_x^{1-0} \psi_0(x)dx,$$

i.e.

$$\phi_0(x) \in K_1 = K.$$

$\diamondsuit$

Appendix B

First, we note that the cosine expansion of Legendre's polynomials of equation (8.911.4) of Gradshteyn and Ryzhik (1994) is not completely accurate. We obtain the following formula:

$$
\begin{aligned}
P_{2n}(\cos\theta) &= 2\cdot\frac{(4n-1)!!}{2^{2n}(2n)!}\left\{\cos 2n\theta + \frac{1}{1}\cdot\frac{2n}{4n-1}\cos(2n-2)\theta\right.\\
&+\cdots+\frac{1\cdot 3\cdots(2k-1)}{1\cdot 2\cdots k}\cdot\frac{2n(2n-1)\cdots(2n-k+1)}{(4n-1)(4n-3)\cdots(4n-2k+1)}\cos 2(n-k)\theta\\
&\left.+\cdots+\left(\frac{1}{2}\right)\frac{1\cdot 3\cdots(2n-1)}{1\cdot 2\cdots n}\cdot\frac{(2n)(2n-1)\cdots(n+1)}{(4n-1)(4n-3)\cdots(2n+1)}\cdot\mathbf{1}\right\};\\
P_{2n+1}(\cos\theta) &= 2\cdot\frac{(4n+1)!!}{2^{2n+1}(2n+1)!}\left\{\cos(2n+1)\theta + \frac{1}{1}\cdot\frac{2n+1}{4n+1}\cos(2n-1)\theta\right.\\
&+\cdots+\frac{1\cdot 3\cdots(2k-1)}{1\cdot 2\cdots k}\cdot\frac{(2n+1)(2n)\cdots(2n-k+1)}{(4n+1)(4n-1)\cdots(4n-2k+1)}\cos(2(n-k)+1)\theta\\
&\left.+\cdots+\frac{1\cdot 3\cdots(2n-1)}{1\cdot 2\cdots n}\cdot\frac{(2n)(2n-1)\cdots(n+1)}{(4n-1)(4n-3)\cdots(2n+1)}\cdot\cos\theta\right\}
\end{aligned}
$$

This can be obtained by the generating function of Legendre's polynomials as in Hobson (1931, page 20).

The cosine expansion yields finite summation while each term only involves integrations of trigonometric functions. One may obtain a closed form expression for b_n's. First, we consider the even terms in the exponential covariance function $e^{-\theta/a}$,

$$
P_{2n}(\cos\theta) = \sum_{k=0}^{n} C_k \cos[2(n-k)\theta],
$$

where

$$
\begin{aligned}
C_0 &= a_0\cdot 1, \quad C_1 = a_0\frac{2n}{4n-1}, \quad \cdots\\
C_k &= a_0\cdot\frac{1\cdot 3\cdots(2k-1)}{1\cdot 2\cdots k}\cdot\frac{2n(2n-1)\cdots(2n-k+1)}{(4n-1)(4n-3)\cdots(4n-2k+1)}, \quad \cdots\\
C_n &= a_0\cdot\frac{1}{2}\frac{1\cdot 3\cdots(2n-1)}{1\cdot 2\cdots n}\cdot\frac{2n(2n-1)\cdots(n+1)}{(4n-1)(4n-3)\cdots(2n+1)},
\end{aligned}
$$

and a_0 is the normalized coefficient with $a_0 = 2\frac{(4n-1)!!}{2^{2n}(2n)!}$. That is, (note there is supposed to have a positive constant in front),

$$
b_{2n} = \int_0^{\pi} e^{-\theta/a} P_{2n}(\cos\theta)\sin\theta d\theta = \sum_{k=0}^{n}\int_0^{\pi} e^{-\theta/a} C_k \cos[2(n-k)\theta]\sin\theta d\theta := \sum_{k=0}^{n} C_k x_k,
$$

where

$$\begin{aligned} x_k &= \int_0^{\pi} e^{-\theta/a} \cos[2(n-k)\theta] \sin\theta d\theta \\ &= (1+e^{-\pi/a}) \left[\frac{2(n-k)+1}{1/a^2+(2(n-k)+1)^2} - \frac{2(n-k)-1}{1/a^2+(2(n-k)-1)^2} \right], \quad k=0,1,\ldots,n. \end{aligned}$$

The odd terms based on the cosine expansion of Legendre's polynomials can be obtained in the same way.

For the spherical covariance function, the calculation of the coefficients b_n based on the cosine expansion of Legendre polynomials could be obtained similarly. For example, the even terms are given by

$$\begin{aligned} b_{2n} &= \int_0^a \left(1 - \frac{3\theta}{2a} + \frac{\theta^3}{2a^3}\right) P_{2n}(\cos\theta) \sin\theta d\theta \\ &= \sum_{k=0}^{n} \int_0^a \left(1 - \frac{3\theta}{2a} + \frac{\theta^3}{2a^3}\right) C_k \cos[2(n-k)\theta] \sin\theta d\theta := \sum_{k=0}^{n} C_k x_k, \end{aligned}$$

where

$$\begin{aligned} x_k &= \frac{1}{2}\left[\frac{1}{2(n-k)+1} - \frac{1}{2(n-k)-1}\right] + \frac{3}{2a^2}\left[\frac{\cos(2(n-k)+1)a}{(2(n-k)+1)^3} - \frac{\cos(2(n-k)-1)a}{(2(n-k)-1)^3}\right] \\ &\quad - \frac{3}{2a^3}\left[\frac{\sin(2(n-k)+1)a}{(2(n-k)+1)^4} - \frac{\sin(2(n-k)-1)a}{(2(n-k)-1)^4}\right], \qquad \text{for } 0 \le k \le n-1, \end{aligned}$$

and

$$x_n = 1 + \frac{3}{a^2}\cos a - \frac{3}{a^3}\sin a.$$

Thus, a closed form expression for b_{2n} is obtained through finite summations.